Drawing Together Euclidean and Topological Threads

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ABSTRACT

The work of Antoni Gaudí (1852–1926), in “returning to the origin” to find originality draws on natural precedent for the relationship of growth and its influence on form. This seems to test the relationship of mathematics and geometry to the living reality of combined second order hyperbolic surfaces more than that of any other architect. The design for the Sagrada Família Church in Barcelona, towards the completion of which he devoted the final phase of his career, is based on assemblies of such surfaces throughout the whole composition of the building: helicoids, hyperbolic paraboloids, and hyperboloids of revolution of one sheet. There are other examples of the use of such forms but these tend to be more singular for example in the work of Felix Candella (b. 1910). This paper explores a number of methods employed in the quest to remodel and resolve the use of these surfaces into a measurable interpretation that can be used to advance the building work.

At the local level and individually these ruled surfaces provide for comparatively simple surface definition and description to building operatives, a clear rationale underlying their choice. At a more global level, simplicity turns to complexity in their mutually intersecting organisation. Second order surfaces intersected in space give fourth order curves. How best might the latter-day successors to the project interpret the physical reality of such surfaces when applied to stone? The options range from Euclidian geometry, to *stereotomically motivated* descriptive geometry, and to haptic engagement assisted by the computer. Reporting through work-in-progress, the dilemmas presented by an interpretation of a 1917 drawing by Gaudí for the Passion Façade suggest a multi-level approach or use of diverse modelling techniques ranging from almost purely visual and haptic engagement to highly controlled use of geometry through programming and parametric design to provide the best overall understanding of this spatial challenge.

***Keywords and phrases*:** architecture, design, geometry, Antoni Gaudí, form, information technology, spatial information, topology

1.0 CONTEXT

Gaudí worked on the Sagrada Família church for forty-three years spanning almost his entire career. The final twelve-year phase of this involvement leading to his death in 1926 is very much the most obscure. Although this was a period of intense design activity devoted exclusively to the church, there was relatively little built outcome during this time compared to the previous three decades. More significantly the modeling work from this period departs completely from what had gone before. During these twelve years Gaudí invented a codex, a design *modus operandi* by which future collaborators could confidently continue work that Gaudí himself would not see built in his lifetime. The codex is based on ruled surfaces in combination, providing for the first time, a 'describable' building, as opposed to one whose construction was based on sketches and day-to-day discussion with the operatives involved (Burry, 1993). Gaudí's surviving modeled and drawn material from that period, poses challenges for researchers considering ruled surfaces from a number of points of view: both intuitive and mathematical. We consider here the case of the hyperbolic paraboloid and the colonnade that protects the sculptures of the prophets from the elements in the Passion Façade of the Sagrada Família Church.

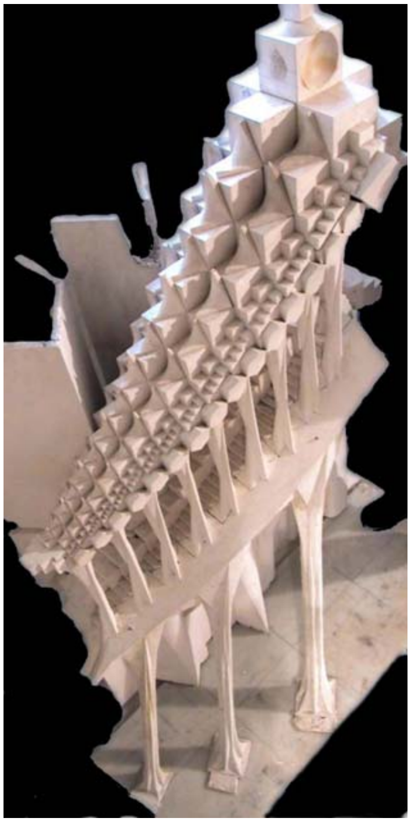

Figure 1. Plaster model of the Colonnade and cornice made by Puig i Boada, Cardoner and assistants during the 1980s at 1:25 scale

Figure 2. Detail of colonnade from 1:25 model

2.0 EVIDENCE

The information for the form and assembly of the colonnade elements resides in a surviving photograph of a drawing of the whole façade in Gaudí's hand completed in 1917 (Tokutoshi, Torii, 1983.) (fig. 3). From this drawing a 1:25 plaster model of one side of the (symmetrical) assembly was completed under the direction of one of Gaudí's former collaborators Puig i Boada during the 1970s, completed by Cardoner and assistants during the 1980s (figs 1 & 2).

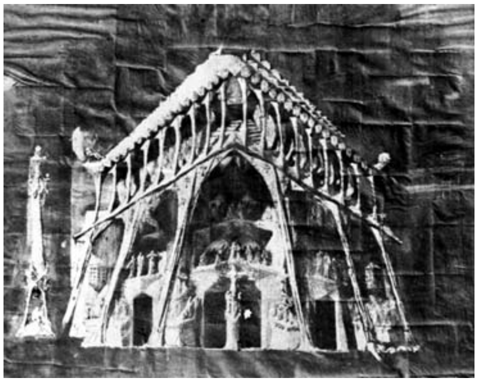

Figure 3. 1917drawing by Gaudí of the Passion Façade

Figure 4. Passion Façade in 2001

The columns of the upper colonnade have been the subject of lengthy and, as yet, unresolved design refinement through reiterative modelling in plaster at larger scale. They are now undergoing a new geometrically led digital generative approach referred to in "The Generative Trail" below

As the time of construction now approaches, a definitive digital model is required for the whole column, cornice assembly leading to detailed design resolution and ultimately 1:1 scale templates and contour diagrams of each stone to communicate with the stone mason.

Current work includes the translation of all earlier plaster modelled work to the closest possible digital facsimile for the purpose of measurement and analysis. The primary source to which all design decision making ultimately refers back, however, is the photograph of the 1917 drawing.

3.0 TRANSLATION FROM PLASTER TO VIRTUAL

Digitising the plaster models is ostensibly simple and essentially a haptic, topological exercise using a 3D digitiser[1] to input the data, the location of nodes and edges of the model in physical space straight into a pre-calibrated coordinate system in proprietary 3D software.

The surface reconstruction work that follows in the virtual domain can be far more demanding, however, taking the user back into a series of mutually impacting local Euclidean systems (Ghiselli-Crippa, T., Hirtle, S. C., & Munro, P., 2001, In press) where a network of linked decisions awaits. The true geometrical intention of the model is now challenged by firstly the modellist's interpretation of the original drawing, secondly the accuracy achieved in the production of the plaster model at the scale of 1:25, thirdly, the hand-eye accuracy in measurement, fourthly, the accuracy of identifying each node with a metal pointer on a soft, deformable medium such as plaster and fifthly, the limits on accuracy in the digitiser's computation of the relative position of its tripartite articulated arm in the established physical coordinate system and its translation to the point in the virtual coordinate system. Clearly the step from 1:25 plaster working to 1:1 digital working greatly magnifies the impact of each small shift in the position of a node when reconstructing the surfaces. Repeatability experiments indicated that the points could be located within a sphere of 1mm radius, that is 25mm at real scale. For a column, which stands approximately 8 metres high, this was acceptable. But this demonstrates in part why the data could not be successfully entered directly as surfaces despite the fact that the software offers this potential.

[1] A digitiser is a machine that is linked to a computer for measuring points in space. The digitiser has a point at the end of a fixed articulated arm which can input information directly within certain 3D modeling software as three dimensional coordinates.

An equally important reason is the divergence of the constructed plaster model from a geometrical 'ideal', the possibility of small tilts and variations in assembling the masses and surfaces to create an overall fit.

Having constructed a digital facsimile of the most current 1:25 model of a single column as a closed surface model, this file was then used to reproduce another physical model in wax through rapid-prototyping. (fig 5)

The digital model could now be used as a reference with which to compare new geometrically synthesized designs for the column or as a "framework" over which a new model controlled by parameters rather than absolute dimensions and relationships could be constructed.

Figure 5. Wax model generated by a rapid prototyping machine at the Sagrada Família from the digital model

4.0 RULED SURFACES: THE HYPERBOLIC PARABOLOID

The most obvious, visually identifiable type of surface in the surface assembly for the column is the hyperbolic paraboloid. The general algebraic equation of degree two for this surface is:

$$x^2/a^2 - y^2/b^2 + z = 0$$

This is the surface that is formed when one of the vertices of a deformable quadrilateral surface is moved out of the plane of the figure. Any four non-coplanar points can define three of these surfaces, appearing as a more or less twisted saddle. Typical examples in nature are the web between the fingers and toes or between tree root and trunk.

Margarit and Buxade's diagrams (Margarit, J. and Buxade, C.: 1969, p. 12) (figs 6 and 7) generate another equation. The coordinates of any point on a finite portion of the surface, the area bound by the lines joining OMNP are given by the equation:

$$z = K.x.y.\sin \omega$$

where:

$$K = f / (a.b.\sin \omega)$$

and

- f *is the perpendicular displacement of the fourth vertex from the plane defined by the other three vertices*
- a *is the length of the boundary edge of the surface on the x axis (referred to above as OM)*
- b *is the length of the boundary edge of the surface on the y axis (referred to above as ON)*

ω *is the angle between the x- and y- axes (or OM and ON above)*

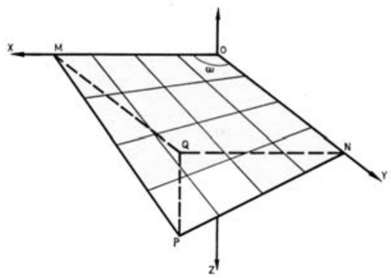

Figure 6. Diagram showing the construction of a bound portion of a hyperbolic paraboloid (taken from Margarit & Buxade)

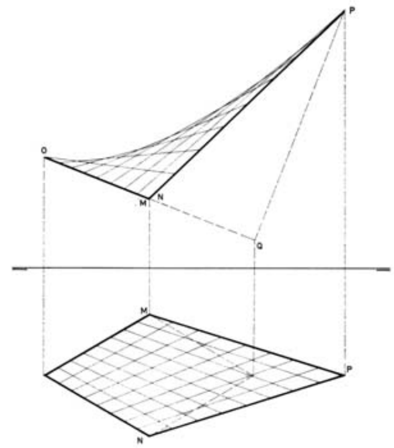

Figure 7. The relationship of the plane and the surface (taken from Margarit & Buxade)

Hyperbolic parabolic surfaces are simple to reproduce graphically where they are bounded, that is where the four edges and vertices of the portion of the surface are explicit.

In the colonnade columns and cornice as elsewhere in the building, however, there are instances where only three or fewer vertices are known. In these instances, the simultaneous solution to the algebraic equations above is at best a line on which the fourth vertex lies. Resolving the surface moves into the domain of the programmer applying a least squares algorithm and, for imprecise data such as this, requires a spread of more than nine points or straight-line-linked pairs of points on the surface.[2]

5.0 UNBOUNDED SURFACES

There are notable examples in Gaudí’s work where the straight-line directrices lying on the surface or edge of the bound surface are exploited in combining elements. Two bordering hyperbolic paraboloids are to be seen meeting on a straight line boundary immediately below the upper colonnade where “branches” from the large tree like columns supporting the canopy over the Crucifixion scene cross to form the central arch rather than meeting in a point in the centre. (Fig 8.) The vaults in the cloistered entrance to the crypt of the Colonia Guëll chapel (1908–15), a project realised without the pressing need for the geometrical rigour that became the focus of the latter years of Gaudí’s work at the Sagrada Familia also exhibit this arrangement of bordering hyperbolic paraboloids applied here with rustic materiality. (fig 9)

[2] Dr Gleb Belikov Deakin University school of Computing and Mathematics is currently engaged in developing tools for this process.

Fig 8. Hyperbolic paraboloid 'branches' overlapping and meeting on their directrices in the Passion Façade

Fig 9. Hyperbolic paraboloids meeting on their directrices at the Colònia Güell Crypt

In contrast, there are three areas in the colonnade where the intersection of the hyperbolic parabolic surface with another geometry results in unbounded surfaces. The first example is at the head of the column where its branch-like lateral hyperbolic parabolic extensions meet a series of hexagonal prismic, possibly basalt[3] lintels spanning between the columns at the front and the wall with niches for the statuary at the back. A second example in the column itself is at its foot where it meets a flight of steps within the colonnade and the sloping canopy of the huge Crucifixion porch below in front of the colonnade. (Fig 10.)

[3] This is a characteristically occurring form in nature. There are hexagonal section columns in the Colonia Guell chapel that employ Basalt prisms in their natural state.

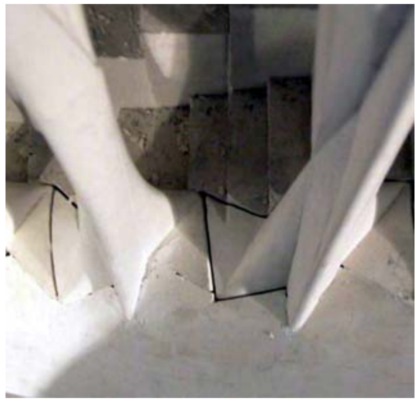

Figure 10. 1:25 plaster model: detail of column feet

The third example is an interesting discovery made in the process of modelling the cornice. The hexagonal prisms referred to above have inclined and dished hexagonal ends facing the front of the cornice. These are the locations for Venetian glass mosaic relief letters of words similar to those seen on the surface of the transept towers. (Figs 11 & 12)

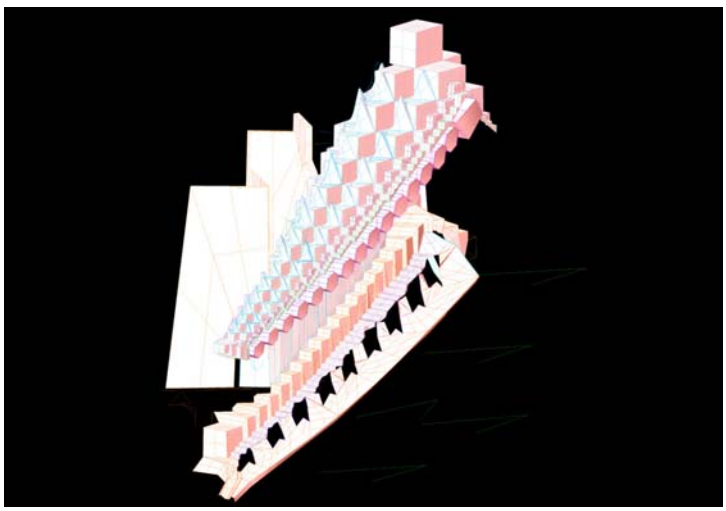

Figure 11. Digitised model of cornice

Figure 12. Dished hexagons

The difficulty of surfacing these hexagonal shapes with their six non co-planar vertices, led to the discovery that here was another hyperbolic parabolic surface. By extending the four of thee six sides that intersected at the top and bottom of the "hexagon" (as viewed from the front) two further points of virtual intersection were established lying to the right and to the left of the figure. Joining these four points gave the boundaries of a hyperbolic parabolic surface. The fifth and sixth sides of the hexagon (left and right apparently vertical) were projected onto this ruled surface and the projected curves used to trim it back to a six-sided figure. (Fig 13)

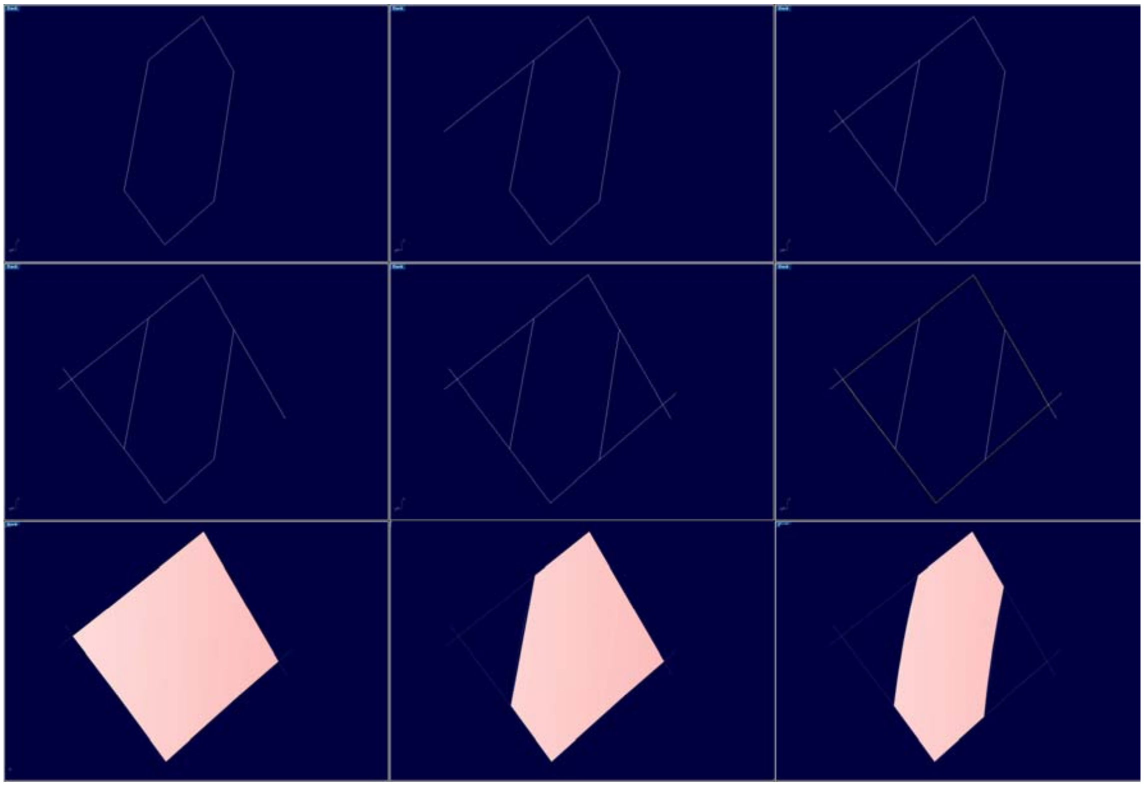

Figure 13. Nine images showing the process of constructing the hyperbolic paraboloid surface of the hexagonal prism ends with virtual boundaries

In the case of the column model, establishing virtual boundaries for the hyperbolic paraboloids 'cut' by the boundary conditions at the head and foot of the column through geometrical or intuitive means proved impossible. In these cases only one vertex (the uppermost point for the 'root' surfaces at the base or the lowest point for the 'branch' surfaces at the top), two partial edges and the points of intersection with the stair or cornice were explicit. This information was insufficient to establish the surface boundaries and hence characteristics through trial and error in classic reiterative graphical design mode. Here the speed and number of iterations attainable by a non human processor were required in conjunction with the more than nine points on the surface needed to apply the least squares algorithm.

In practice, to produce the closed surface model of the column in the time and with the means to hand such indeterminate surfaces were constructed from a patchwork of pieces, defined by curves digitised across the surface using the streaming data command to draw a degree 3 curve between a close array of points (1–2mm intervals) in addition to the ruled line partial boundaries. The resulting curves were rebuilt a number of times to achieve a sufficiently smooth edge to construct neighbouring surfaces. The patchwork surfaces sufficed for the purpose of producing the wax 3D print but are less than elegant on close inspection.

6.0 THE GENERATIVE TRAIL

There are other leads in the quest to discover the true column. While the work described above is largely of a forsenic and analytical nature, abstract thinking about the form of the column and the problem of the unbounded hyperbolic paraboloid has given rise to another line of enquiry. Together Mark Burry, Gleb Belikov, School of Mathematics and Computing, Deakin University and Peter Wood, computer programmer in Wellington New Zealand have designed a tool to generate hyperbolic parabolic surfaces, intersecting with a central hyperboloid. The program can generate the form from the limited information of the parameters of the hyperboloid, the planes of intersection at its head and base, one boundary point for the hyperbolic paraboloid and the two points of intersection between the hyperboloid, the hyperbolic paraboloid and the intersecting plane. This is in effect a specialized parametric design tool that can produce any number of visually and formally dissimilar columns all belonging to the same geometrical family. Early results when compared with the original drawing and considered in the light of Gaudi's wider work are very promising.

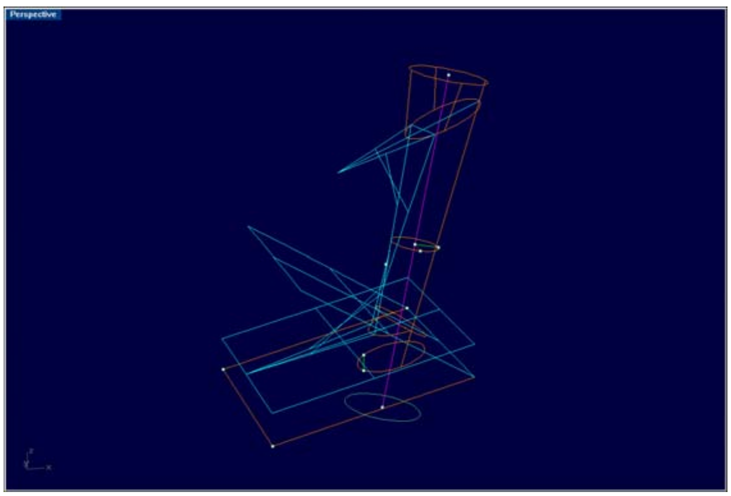

Figure 14. Wireframe of a first test of the program to combine hyperboloids with hyperbolic paraboloids

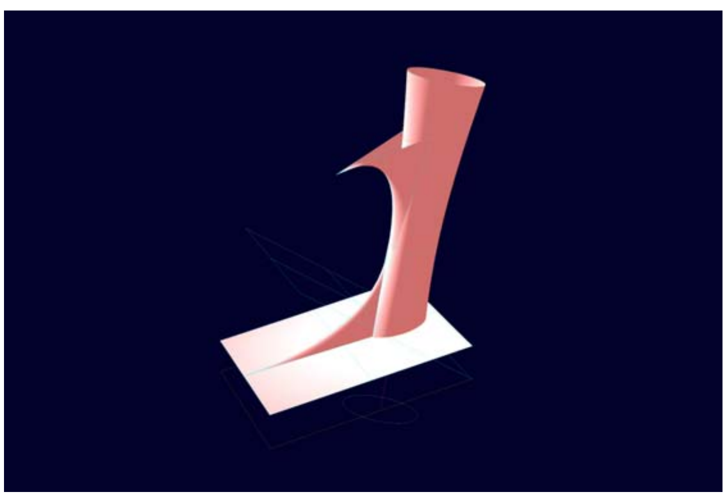

Figure 15. Render of a first test of the program to combine hyperboloids with hyperbolic paraboloids

7.0 TOPOLOGY OR EUCLIDEAN ABSTRACTION

This paper reports on work in progress whose conclusion is anticipated in September 2003: work of navigation through complex existing physical and representational spatial information with a view to producing a model of great precision that resolves conflicts and dilemmas in a way true to the clearly codified intentions of the original architect. The individual constituent forms which are the currency of this codex can, by their nature, be simply described both mathematically and topologically. The architectural resolution of an assembly of many such forms introduces complexity.

The post-digital era gives preference neither to a purely mathematical nor a purely haptic graphical approach to assimilating and applying this information but advances both causes. Similarly our understanding of this complex architectural terrain is assisted through both abstract mathematical consideration and a more physical engagement. With respect to Gaudí himself, it seems unlikely that he would have taken the purely mathematical approach and equally unlikely that he would reject the opportunities of best-fit algorithms and other approaches to optimisation enquiry now available. Parametric design software has, to date, proved a significantly useful companion and successor to the slow and laborious process of manipulating parameters in the physical/graphical domain of the plaster model workshop and the drawing board. Its opportunities continue to emerge with continued exploration.

In its diversity, this process draws parallels with current theory of spatial cognition and daily reinforces for the participants the richness and depth of the original creative mind.

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